MATH 5A - SAMPLE FINAL EXAM

(1) Find the following limits if they exist. If not, why not?

(a) $\lim\limits_{x\to 0^-} \frac{|x|}{x}\cos x$ (b) $\lim\limits_{x\to 2^-} \frac{1-x}{x-2}$ (c) $\lim\limits_{x\to\infty} \frac{x-3}{x^2}$ $\frac{1}{x^2}$ $\frac{1}{x^2}$

If $x\to 0^-$ then $x<0$
so $|x|=-x$

$\lim\limits_{x\to 0^-} \frac{-x}{x}\cos x$

$=\lim\limits_{x\to 0^-}(-\cos x)$

$=\boxed{-1}$

" $\frac{-1}{0}$ " $\to\infty$ $\to\infty$
vertical asymptote
So $+\infty$ or $-\infty$?

$\frac{-}{-}=+$

$\boxed{\infty}$

$=\lim\limits_{x\to\infty} \frac{\frac{1}{x}-\frac{3}{x^2}}{1}$ $\to 0$ $\to 0$ $=\boxed{0}$

(2) Use the difference quotient and definition of derivative to find f'(x) if f(x) = x^3 - x.

$f'(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\frac{(x+h)^3-(x+h)-(x^3-x)}{h}$

$=\lim\limits_{h\to 0}\frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{x}-h-\cancel{x^3}+\cancel{x}}{h}$

$=\lim\limits_{h\to 0}\frac{\cancel{h}(3x^2+3xh+h^2-1)}{\cancel{h}}=\boxed{3x^2-1}$

OR

Since $1+\tan^2 x=\sec^2 x$
$h(x)=\sec^6 x$ so $h'(x)=6\sec^5 x\sec x\tan x$
$=6\sec^6 x\tan x$

(3) Find the derivative of each of the following functions and simplify your answer:

(a) f(x) = $\sqrt{x}\,(x^2+2)$ (b) h(x) = (1 + $\tan^2$x)^3 (c) g(x) = $\frac{x}{\sqrt{x^2+1}}=x(x^2+1)^{-1/2}$

$f(x)=x^{5/2}+2x^{1/2}$

$f'(x)=\frac{5}{2}x^{3/2}+x^{-1/2}$

$f'(x)=\frac{1}{2}x^{-1/2}(5x^2+2)$

a $\boxed{f'(x)=\frac{5x^2+2}{2\sqrt{x}}}$

$h'(x)=3(1+\tan^2 x)^2\frac{d}{dx}(1+\tan^2 x)$

$h'(x)=3(1+\tan^2 x)^2(2\tan x\frac{d}{dx}\tan x)$

b $\boxed{h'(x)=6(1+\tan^2 x)^2\tan x\sec^2 x}$

$g'(x)=(x^2+1)^{-1/2}-\frac{1}{2}x(x^2+1)^{-3/2}(2x)$

$g'(x)=(x^2+1)^{-3/2}(x^2+1-x^2)$

c $\boxed{g'(x)=\frac{1}{(x^2+1)^{3/2}}}$

(4) Find the y-intercept of the line tangent to the curve $x^2 - xy - y^2 = 1$ at (2,1)

Slope of tangent: $\left.\frac{dy}{dx}\right|_{(2,1)}$ ⇒ Need implicit differentiation

$\frac{d}{dx}(x^2 - xy - y^2) = \frac{d}{dx}1$

$2x - y - x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx}(-x-2y) = y - 2x$

$\frac{dy}{dx} = \frac{y-2x}{-x-2y}$

$m = \left.\frac{dy}{dx}\right|_{(2,1)} = \frac{-3}{-4} = \frac{3}{4}$

Eqn of line

$y - 1 = \frac{3}{4}(x-2)$

$y = \frac{3}{4}x - \frac{1}{2} \Rightarrow \boxed{-\frac{1}{2}}$

(5) Integrate:

(a) $\int_0^{\pi/4} \sin x \cos^3 x \, dx$

$u = \cos x$

$du = -\sin x \, dx$

$-\int_1^{\sqrt{2}/2} u^3 \, du$

$= -\frac{1}{4}u^4 \Big]_1^{\frac{\sqrt{2}}{2}}$

$= -\frac{1}{4}\left(\left(\frac{\sqrt{2}}{2}\right)^4 - 1\right)$

$= -\frac{1}{4}\left(\frac{1}{4} - 1\right)$

$= -\frac{1}{4}\left(-\frac{3}{4}\right)$

$= \boxed{\frac{3}{16}}$

(b) $\int_0^2 (3-x)^2 \, dx$

$u = 3 - x$

$du = -dx$

$-\int_3^1 u^2 \, du$

$= -\frac{1}{3}u^3\Big]_3^1$

$= -\frac{1}{3}(1 - 27)$

$\boxed{\frac{26}{3}}$

(c) $\int \frac{x^3}{\sqrt{x^2 - 1}} dx$

$u = x^2 - 1$ $\quad x^2 = u + 1$

$du = 2x\,dx$

$= \int \frac{x^2}{\sqrt{x^2-1}} x\,dx$

$= \int \frac{u+1}{\sqrt{u}} \frac{du}{2}$

$= \frac{1}{2}\int (u^{1/2} + u^{-1/2})\,du$

$\frac{1}{3}u^{3/2} + u^{1/2}$

$\boxed{\frac{1}{3}(x^2-1)^{3/2} + (x^2-1)^{1/2} + C}$

(6) Given $f(x) = x(1-x)^{2/5}$,

(a) find the interval(s) on which the function f is

(i) increasing (ii) decreasing (iii) concave up (iv) concave down

(b) find all critical points (c) inflection points (d) find all extrema

(e) given the above information, sketch a graph of the above function.

Inc: $(-\infty, \frac{5}{7})$ $(1, \infty)$

dec $(5/7, 1)$

Local max at $(\frac{5}{7}, f(\frac{5}{7}))$ $\approx .43$

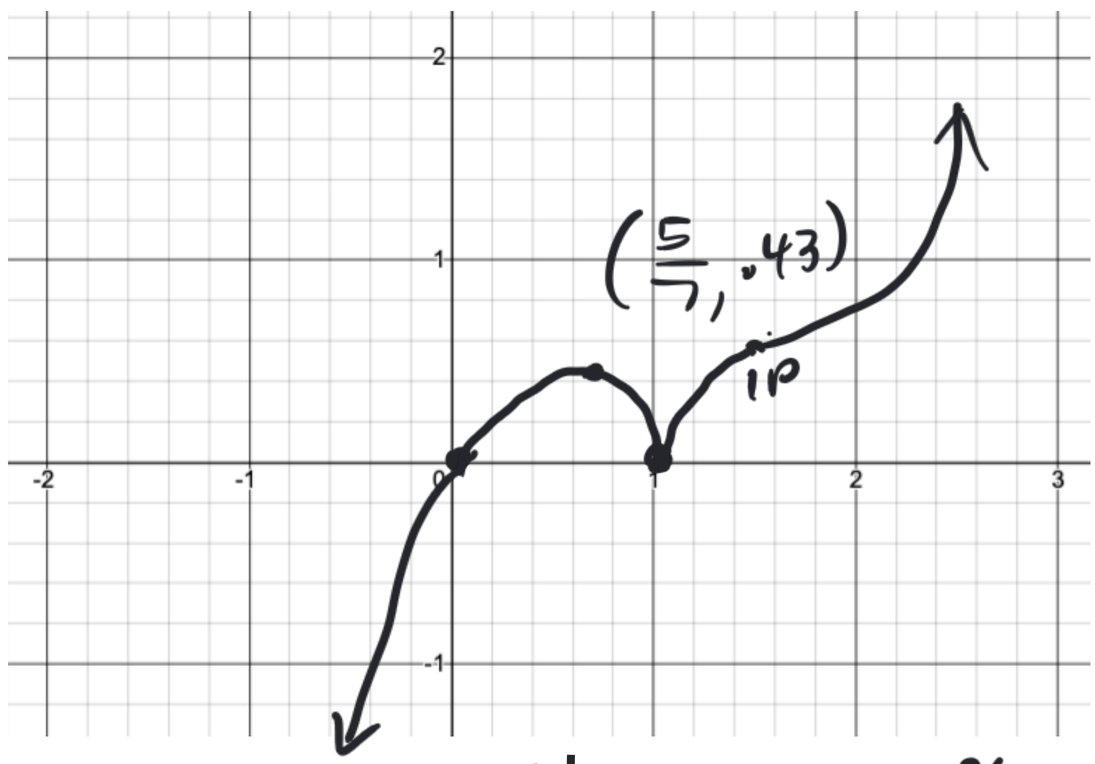
Local min at $(1, 0)$

CU $(\frac{10}{7}, \infty)$

CD $(-\infty, 1)$ $(1, \frac{10}{7})$

Inflection points

$(1, 0)$ $(\frac{10}{7}, 1.01)$



$\boxed{f(x)}$

$f(x) = x(1-x)^{2/5}$

x-int: 0, 1

y-int: 0

end behavior:

$\lim_{x\to\infty} f(x) = \infty$

$\lim_{x\to-\infty} f(x) = -\infty$

$\boxed{f'(x)} = (1-x)^{2/5} + \frac{2}{5}x(1-x)^{-3/5}(-1)$

$f'(x) = \frac{1}{5}(1-x)^{-3/5}(5(1-x) - 2x)$

$$\boxed{f'(x) = \frac{5-7x}{5(1-x)^{3/5}}}$$

crit #s $5/7, 1$

$+ \quad 5/7 \quad - \quad 1 \quad +$

$f''(x)$

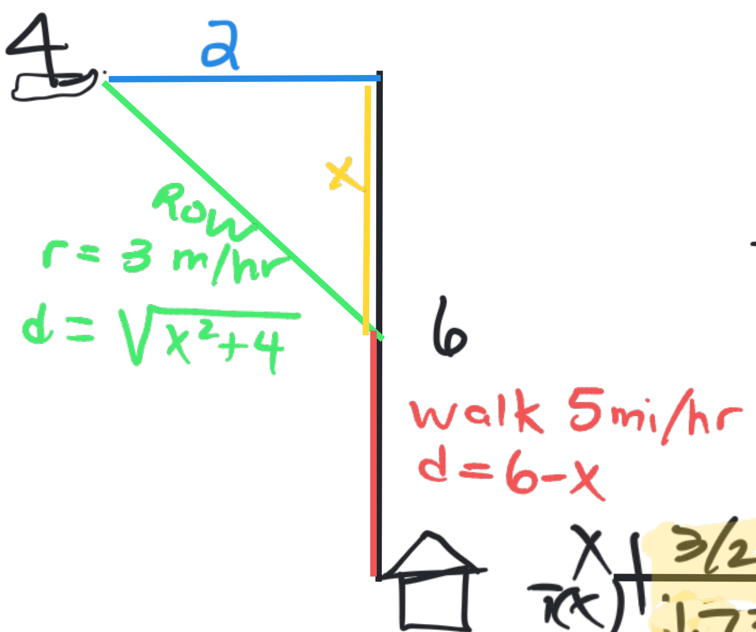
$- \quad 1 \quad - \quad \frac{10}{7} \quad +$

$f''(x) = \frac{1}{5}\frac{d}{dx}(5-7x)(1-x)^{-3/5} = \frac{1}{5}\left[-7(1-x)^{-3/5} + (5-7x)(\frac{-3}{5})(1-x)^{-8/5}(-1)\right]$

$= \frac{1}{5}\left[-7(1-x)^{-3/5} + \frac{3}{5}(5-7x)(1-x)^{-8/5}\right]$

$= \frac{1}{5}\cdot\frac{1}{5}(1-x)^{-8/5}(-35(1-x) + 3(5-7x)) = \boxed{\frac{-20+14x}{25(1-x)^{8/5}}}$

(7) A person in a rowboat 2 miles from the nearest point on a straight shoreline wishes to reach a house 6 miles farther down the shore. If the person can row at a rate of 3 mi/hr and walk at a rate of 5mi/hr. find the least amount of time required to reach the house. (Show all steps you used to determine minimum is absolute)



Minimize Time = Time Row + Time Walk

$= \frac{\text{dist row}}{\text{rate row}} + \frac{\text{dist walk}}{\text{rate walk}}$

$= \frac{\sqrt{x^2+4}}{3} + \frac{6-x}{5}$

$T(x) = \frac{1}{3}\sqrt{x^2+4} - \frac{1}{5}x + \frac{6}{5} \quad 0 \le x \le 6$

$T'(x) = \frac{1}{6}(x^2+4)^{-1/2}\,2x - \frac{1}{5} = \frac{x}{3\sqrt{x^2+4}} - \frac{1}{5}$

$T'(x) = 0 \Rightarrow \frac{x}{3\sqrt{x^2+4}} = \frac{1}{5}$

$5x = 3\sqrt{x^2+4}$

$25x^2 = 9(x^2+4)$

$16x^2 = 36$

$x = \frac{6}{4} = \frac{3}{2}$

x	3/2	0	6
T(x)	1.73	1.87	2.11

MIN TIME $\frac{52}{30} = 1.73$ hrs

(8) Find the absolute min/max of f(x) = x-2cosx on the interval [-π,π].

closed interval method

crit #s: $f'(x) = 1 + 2\sin x$

$f'(x) = 0 \Rightarrow \sin x = -1/2$

$x = -\frac{5\pi}{6}, -\frac{\pi}{6}$

x	$-5\pi/6$	$-\frac{\pi}{6}$	$-\pi$	π
f(x)	-.89	-2.3	-1.1	5.14
		MIN		MAX

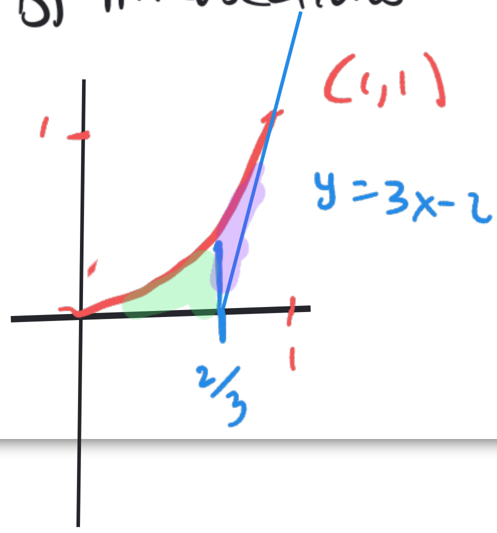
(9) (a). Find the tangent line to y = x^3 , when x=1. $f'(x) = 3x^2 \quad f(1) = 1$

(b) Find the area between the line from part (a), the graph of y = x^3 and the x axis, in the first quadrant.

a) $m = f'(1) = 3$

tangent line $y - 1 = 3(x-1) \Rightarrow y = 3x - 2$

b) Intersections



Area $\int_0^{2/3} x^3\,dx + \int_{2/3}^{1} (x^3 - (3x-2))\,dx$

$\left.\frac{x^4}{4}\right]_0^{2/3}$

$\frac{4}{81}$

$\boxed{\frac{1}{12}}$

$\left[\frac{x^4}{4} - \frac{3}{2}x^2 + 2x\right]_{2/3}^{1}$

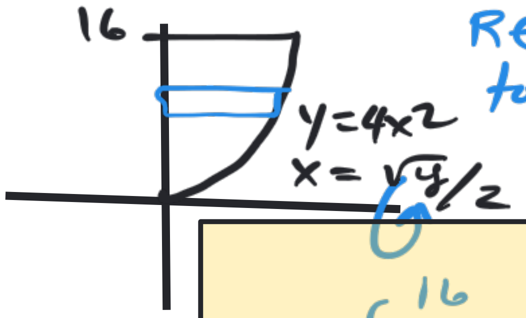
$\frac{1}{4} - \frac{3}{2} + 2 - \left(\frac{4}{81} - \frac{2}{3} + \frac{4}{3}\right)$

(10) Find the volume of the solid resulting when the region in the first quadrant bounded by the graphs of

$y = 4x^2$ and $y = 16$ is revolved about the x-axis.

** SET UP ONLY - TWO WAYS**

(a) cylindrical shells

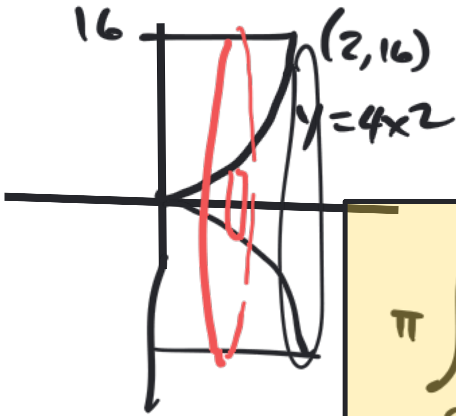


Rectangle parallel to axis $\Rightarrow dy$

"$2\pi r h\, dy$"

$$\boxed{2\pi \int_0^{16} y \frac{\sqrt{y}}{2}\, dy}$$

(b) disks/washers



Slice perp to x-axis $\Rightarrow dx$

$\pi(R_o^2 - R_i^2)\,dx$

$$\boxed{\pi \int_0^{2} \left(16^2 - (4x^2)^2\right) dx}$$

(11) A balloon is rising vertically over a point A on the ground at a rate of 15 ft/sec. A point B on the ground is level with A and is 30 ft. from A. When the balloon is 40 ft. above A, at what rate is its distance from B changing?

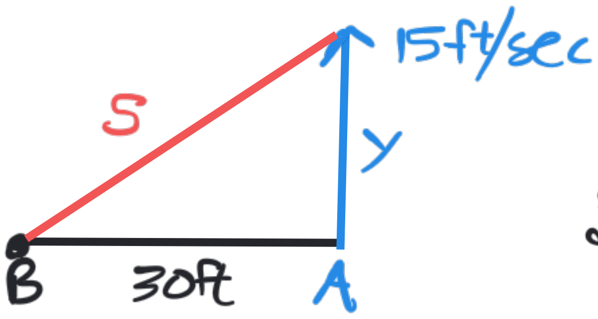
Know

$\frac{dy}{dt} = 15$

want

$\left.\frac{ds}{dt}\right|_{y=40}$

relate s and y



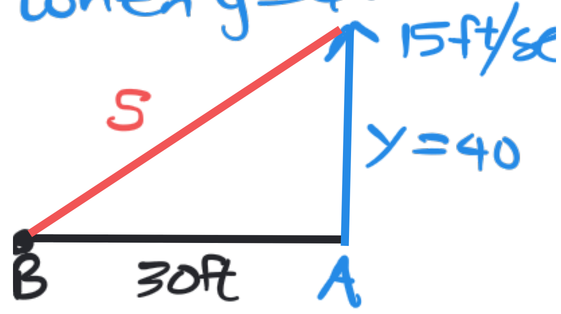
$\frac{d}{dt}$

$$y^2 + 900 = s^2$$

$$2y\frac{dy}{dt} = 2s\frac{ds}{dt}$$

$$\frac{ds}{dt} = \frac{y}{s}\frac{dy}{dt}$$

when y=40



$s^2 = 30^2 + 40^2$

$s = 50$

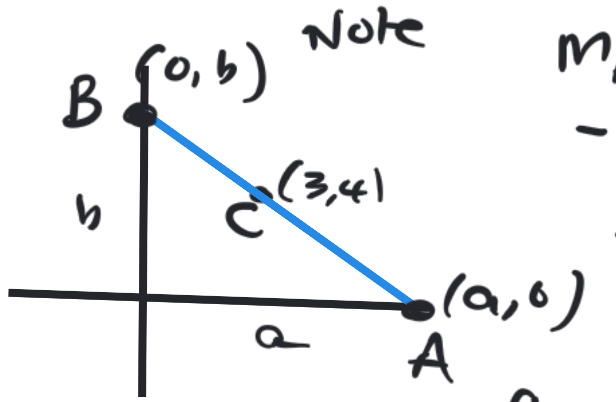
When $y = 40$

$$\left.\frac{ds}{dt}\right|_{y=40} = \frac{40}{50}(15) = \boxed{12\text{ ft/sec}}$$

* Line thru (0,8)/(3,4) $m=-\frac{4}{3}$ $\boxed{y=-\frac{4}{3}x+8}$

(12) Find the equation of the line through (3,4) which cuts from the first quadrant a triangle of minimum area.

Minimize $A=\frac{1}{2}ab$. Relate a & b.



Note $m_{AB}=m_{BC}=m_{AC}$

$-\frac{b}{a}=\frac{4-b}{3} \Rightarrow -3b=a(4-b)$

$a=-\frac{3b}{4-b}=\frac{3b}{b-4}$

so $A=\frac{1}{2}\left(\frac{3b}{b-4}\right)b=\frac{3}{2}\frac{b^2}{b-4}$ $(b>4)$

Crit #s: $A'=\frac{3}{2}\left(\frac{(b-4)2b-b^2}{(b-4)^2}\right)=\frac{3}{2}\left(\frac{b^2-8b}{(b-4)^2}\right)$

$A'=0 \Rightarrow b^2-8b=0$ $b=0, 8$

Since physically, there must be a Δ of min area, it corresponds to $b=8$. See above *

(13) Does the Mean Value Theorem apply to the given function? If so, find "c". If not, why not?

f(x) = $\sqrt{2x+1}$, [0,4]

- f(x) conts for $x \geq -1/2$ so conts [0,4]
- $f'(x)=\frac{1}{\sqrt{2x+1}} \Rightarrow$ f diffable for $x>-1/2$ so diffable (0,4)

MVT applies, find c such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$f'(c)=\frac{f(4)-f(0)}{4-0}$$

$$\frac{1}{\sqrt{2c+1}}=\frac{3-1}{4}=\frac{1}{2}$$

$$\sqrt{2c+1}=2$$

$$2c+1=4$$

$$2c=3$$

$\boxed{c=3/2}$ $\in(0,4)$ ✓